Linear Programming - example taken from Gilbert Strang
***Introduction to Applied Mathematics* Wellesley-Cambridge Press 1986**

An example of **linear programming** problem is to maximize objective function

$$z = 5x_1 + 4x_2 + 3x_3 \tag{1}$$

with unknown variables $x_1, x_2, x_3 \geq 0$ subject to conditions

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 & \leq & 5 \\ 4x_1 + x_2 + 2x_3 & \leq & 11 \\ 3x_1 + 4x_2 + 2x_3 & \leq & 8 \end{array} \tag{2}$$

To avoid system of inequalities in the original problem, new variables can be introduced appropriately and the task can be reformulated as to maximize objective function

$$z = 5x_1 + 4x_2 + 3x_3 \tag{3}$$

with unknown variables $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ subject to conditions expressed in system of equations

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 + x_4 & = & 5 \\ 4x_1 + x_2 + 2x_3 + x_5 & = & 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 & = & 8 \end{array} \tag{4}$$

In other words, the goal of LP is to find such a solution $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ of system of equations (4) which gives maximum value to the objective function $z = 5x_1 + 4x_2 + 3x_3$.

The system of linear equations (4) has many possible solutions for which $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. There are 3 equations and 6 variables in the system. In particular, without taking into account the goal of maximizing the objective function z, any 3 variables among $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ can be arbitrarily chosen as equal to zero, and the system (4) can be converted into a 3×3 system of linear equations for which solution for remaining 3 variables can be obtained.

In the example letting any 3 variables $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ to become zeros, and solving a 3×3 systems of linear equations for remaining variables represents finding out corners or intersection points of 6 planes in six-dimensional space. These points will give different values of the objective function $z = 5x_1 + 4x_2 + 3x_3$. For at least one of the corner points the objective function z will have its maximum value. This corner point will represent the solution of the LP problem under consideration. Out of 6 variables the first one can be arbitrarily chosen in 6 ways, the second one can be chosen in 5 ways, the third one can be chosen in 4 ways what gives $6*5*4 = 6!/(6-3)!$ ordered sequences contaning 3 different variables. The sequence of 3 different elements is ordered in $3*2*1 = 3!$ ways (because on the first place of a 3-element sequence there can be one of three elements, on the second place of the sequence there can be one of two elements, and on the third place of a sequence there can be one of the remaining only one element). In this way 3! of sequences of 3 given elements represent just one *set* of 3 elements where the order of elements does not matter. Therefore among $6!/(6-3)!$ sequences there is only $6!/((6-3)!*3!)$ different sets containing 3 different variables. The total number of sets of three different variables is

$$\frac{6!}{(6-3)!*3!} = \frac{6*5*4*3!}{3!*3!} = \frac{6*5*4}{3!} = \frac{6*5*4}{3*2*1} = 2*5*2 = 20 \qquad (5)$$

There can be 20 of 3×3 systems of linear equations for each set of 3 variables with remaining 3 variables equal to zero. Solution of each of the 20 systems of linear equations 3×3 with remaining 3 variables equal to zero will represent a point of intersection of 6 planes in 6-dimensional space. For each of the 20 intersection points the value of objective function z can be computed. For at least one of the 20 points the objective function z will have its maximum value. The coordinates of the point for which the objective function z has its maximum value represent the solution of the LP problem under consideration.

Fortunately, there is no need to build and solve all of the 20 possible 3×3 systems of linear equations to find the optimal solution of the LP problem. There is a way of finding the optimal solution of a given LP problem just by solving a number significantly lower than 20 of 3×3 systems of linear equations. At the beginning there is a need to start with an arbitrary 3×3 system of linear equations out of possible 20 of 3×3 linear systems. It is done just by choosing arbitrary 3 variables as zeros and solving a 3×3 system of linear equations for the remaining 3 variables. The variables, for which the 3×3 system of linear equations is being solved are called **basic variables**; the remaining variables which are equal to zero, are called **nonbasic variables**.

Just by inspecting the system of linear equations (4) one can see that at the beginning it is convenient to choose x_4, x_5, x_6 as **basic variables** and x_1, x_2, x_3 as **nonbasic variables**, because if $x_1 = x_2 = x_3 = 0$, then the system (4) really becomes a 3×3 system of linear equations with 3 unknown variables x_4, x_5, x_6 and it can be solved immediately with $x_4 = 5, x_5 = 11, x_6 = 8$. The point with coordinates $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{5}, \mathbf{11}, \mathbf{8})$ in 6-dimensional space represents one of possible solutions or in other words **feasible solutions** of the linear system (4) with variables $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ as required.

In a general LP problem the goal is to maximize $\mathbf{cx}$ subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

All of n variables $x_1, x_2, \ldots, x_n$ can be represented as a column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \tag{6}$$

The objective function $\mathbf{cx}$ can be written in vector notation as

$$\mathbf{cx} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \tag{7}$$

what represents

$$\mathbf{cx} = c_1 x_1 \, + \, c_2 x_2 \, + \, \ldots \, + \, c_n x_n \tag{8}$$

where $c_1, c_2, \ldots, c_n$ are constant coefficients of the objective function, and $x_1, x_2, \ldots, x_n$ are values of appropriate variables to be computed in the process of solving LP problem.

In the example $z = \mathbf{cx}$ with row vector of coefficients $\mathbf{c} = [5\ 4\ 3\ 0\ 0\ 0]$

and

$$\mathbf{cx} = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{9}$$

what leads to

$$\mathbf{cx} = 5x_1 \, + \, 4x_2 \, + \, 3x_3 \tag{10}$$

The matrix equation $\mathbf{Ax} = \mathbf{b}$ in general is the system of m linear equations representing m linear constraints involving n variables $x_1, x_2, \ldots, x_n$. The number of constraints m is less than the number of variables n

$$m < n \tag{11}$$

and the linear system $\mathbf{Ax} = \mathbf{b}$ consists of m linear equations in n variables what makes it underdetermined.

The constraints (4) in the exmple can be presented in matrix notation as

$$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{12}$$

with

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{14}$$

and with

$$\mathbf{b} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{15}$$

It is worth noticing that the order of variables in the linear system (4) does not matter, and instead of the originally written system

$$\begin{array}{lllllll} 2x_1 + 3x_2 + x_3 + x_4 & & & = & 5 \\ 4x_1 + x_2 + 2x_3 & + x_5 & & = & 11 \\ 3x_1 + 4x_2 + 2x_3 & & + x_6 & = & 8 \end{array} \tag{16}$$

it can be changed e.g. to read

$$\begin{array}{lllllll} 2x_1 + x_3 + 3x_2 + x_4 & & & = & 5 \\ 4x_1 + 2x_3 + x_2 & + x_5 & & = & 11 \\ 3x_1 + 2x_3 + 4x_2 & & + x_6 & = & 8 \end{array} \tag{17}$$

where column for variable x_3 has been placed before column for variable x_2. In matrix notation the system (17) is

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{18}$$

In particular, after choosing **basic variables** and **nonbasic variables** in the system (4) as described earlier the columns for **basic variables** can be moved ahead of columns for **nonbasic variables** what leads from the original system of equations

$$\begin{array}{rrrrrrl} 2x_1 + & 3x_2 + & x_3 + & x_4 & & & = 5 \\ 4x_1 + & x_2 + & 2x_3 & & + x_5 & & = 11 \\ 3x_1 + & 4x_2 + & 2x_3 & & & + x_6 & = 8 \end{array} \tag{19}$$

to equivalent system of equations

$$\begin{array}{rrrrrrl} x_4 & & & + 2x_1 + & 3x_2 + & x_3 & = 5 \\ & x_5 & & + 4x_1 + & x_2 + & 2x_3 & = 11 \\ & & x_6 & + 3x_1 + & 4x_2 + & 2x_3 & = 8 \end{array} \tag{20}$$

which in matrix notation is written as

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{21}$$

x_4, x_5, x_6 have been chosen as **basic variables**, and x_1, x_2, x_3 have been chosen as **nonbasic variables** as it was mentioned earlier. It is worth noticing that now the vector of variables $\mathbf{x}$ has the **basic variables** listed ahead of **nonbasic variables**

$$\mathbf{x} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{22}$$

Here is another view on the original system of linear equations (4) and on moving its columns corresponding to appropriate variables. The original system of equations

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 + x_4 & & = 5 \\ 4x_1 + x_2 + 2x_3 + x_5 & & = 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 & & = 8 \end{array} \tag{23}$$

can be rewritten in matrix notation as

$$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{24}$$

After performing matrix multiplication $\mathbf{Ax}$ we get

$$\begin{bmatrix} 2 \cdot x_1 + 3 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6 \\ 4 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + 0 \cdot x_6 \\ 3 \cdot x_1 + 4 \cdot x_2 + 2 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{25}$$

There are vectors on both sides of the above equation. The vector on the left hand side of equation (25) has its row components in form of sums and it is therefore equivalent to

$$\begin{bmatrix} 2 \cdot x_1 \\ 4 \cdot x_1 \\ 3 \cdot x_1 \end{bmatrix} + \begin{bmatrix} 3 \cdot x_2 \\ 1 \cdot x_2 \\ 4 \cdot x_2 \end{bmatrix} + \begin{bmatrix} 1 \cdot x_3 \\ 2 \cdot x_3 \\ 2 \cdot x_3 \end{bmatrix} + \begin{bmatrix} 1 \cdot x_4 \\ 0 \cdot x_4 \\ 0 \cdot x_4 \end{bmatrix} + \begin{bmatrix} 0 \cdot x_5 \\ 1 \cdot x_5 \\ 0 \cdot x_5 \end{bmatrix} + \begin{bmatrix} 0 \cdot x_6 \\ 0 \cdot x_6 \\ 1 \cdot x_6 \end{bmatrix} = \\ = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{26}$$

and therefore equivalent to

$$\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_6 =$$

$$= \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \quad (27)$$

and in particular equivalent to

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_6 \right) +$$

$$+ \left(\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x_3 \right) =$$

$$= \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \quad (28)$$

what can be presented as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \quad (29)$$

or presented as in equation (21).

It is now clearly visible that the order of sum components in (27) can be changed, and because of this new order a new rearranged vector $\mathbf{Ax}$ and thus new matrix $\mathbf{A}$ will be obtained, and a new vector $\mathbf{x}$ will have the same variables $x_1, x_2, x_3, x_4, x_5, x_6$ however occurring in a new and appropriate order. In this way matrix columns for **basic** variables will be moved ahead of matrix columns for **nonbasic** variables and a new matrix $\mathbf{A}$ will be built; **basic** variables will be appropriately moved ahead of **nonbasic** variables and the vector of variables $\mathbf{x}$ will be constructed as previously shown with variables in the sequence $x_4, x_5, x_6, x_1, x_2, x_3$.

In general, the vector of variables $\mathbf{x}$ contains parts consisting of **basic** and **nonbasic** variables $\mathbf{x_B}$ and $\mathbf{x_N}$, respectively

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} \quad (30)$$

In the example

$$\mathbf{x_B} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{31}$$

and

$$\mathbf{x_N} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{32}$$

The matrix **A** from equation (21)

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{bmatrix} \tag{33}$$

can be partitioned into a square 3×3 matrix for **basic variables** and $3 \times (6-3)$ matrix for **nonbasic variables**

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right] \tag{34}$$

In a general case of LP problem with m linear constraints and n variables the matrix **A** from equation $\mathbf{Ax} = \mathbf{b}$ can be partitioned into a square $m \times m$ matrix **B** for **basic variables** and $m \times (n-m)$ matrix **N** for **nonbasic variables**

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{N} \end{array}\right] \tag{35}$$

In the example the matrix **B** for **basic variables** is

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

and the matrix **N** for **nonbasic variables** is

$$\mathbf{N} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \tag{37}$$

The first step of finding a solution to LP problem with m constraints begins with choosing m **basic variables** in vector $\mathbf{x_B}$ arbitrarily, and setting the remaining $n - m$ **nonbasic variables** in vector $\mathbf{x_N}$ equal to zeros

$$\mathbf{x_N} = \mathbf{0} \tag{38}$$

Then the matrix $\mathbf{A}$ from linear constraints equation $\mathbf{Ax} = \mathbf{b}$ shall be partitioned into a square $m \times m$ matrix $\mathbf{B}$ multiplying vector $\mathbf{x_B}$ of **basic variables**, and $m \times (n - m)$ matrix $\mathbf{N}$ multiplying vector $\mathbf{x_N}$ of **nonbasic variables**. The linear system of constraining equations $\mathbf{Ax} = \mathbf{b}$ can be now written in the following form involving matrices $\mathbf{B}$ and $\mathbf{N}$, and vectors $\mathbf{x_B}$ and $\mathbf{x_N} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{B} & \mathbf{N} \end{array} \right] \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \left[\mathbf{b} \right] \tag{39}$$

Equation (39) is just a short way of writing

$$\mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{b} \tag{40}$$

Because it has been chosen that $\mathbf{x_N} = \mathbf{0}$ we have $\mathbf{Nx_N} = \mathbf{0}$ what leads to the set of linear system $m \times m$ with m unknown variables in vector $\mathbf{x_B}$

$$\mathbf{Bx_B} = \mathbf{b} \tag{41}$$

The equation (41) can be solved by multiplying its both sides by $\mathbf{B^{-1}}$ from the left, where $\mathbf{B^{-1}}$ is the inverse of the matrix $\mathbf{B}$

$$\mathbf{B^{-1}Bx_B} = \mathbf{B^{-1}b} \tag{42}$$

$\mathbf{B^{-1}B} = \mathbf{I}$ where $\mathbf{I}$ is the unit matrix and therefrom the solution of the linear system (41) is

$$\mathbf{x_B} = \mathbf{B^{-1}b} \tag{43}$$

The objective function also known under the name of **cost** function $\mathbf{cx}$ in vector notation expressed with **basic** and **nonbasic variables** can be computed as

$$\mathbf{cx} = \left[\begin{array}{c|c} \mathbf{c_B} & \mathbf{c_N} \end{array} \right] \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} \tag{44}$$

where $\mathbf{c_B}$ and $\mathbf{c_N}$ represent the constant coefficients corresponding to **basic** and **nonbasic variables**, respectively what leads to

$$\mathbf{cx} = \mathbf{c_B x_B} + \mathbf{c_N x_N} \tag{45}$$

With $\mathbf{x_N} = \mathbf{0}$ the cost function can be computed as

$$\mathbf{cx} = \mathbf{c_B x_B} = \mathbf{c_B B^{-1} b} \tag{46}$$

With $\mathbf{x_N} = \mathbf{0}$ the equation (39) can be rewritten as

$$\mathbf{Ax} = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{N} \end{array}\right] \left[\begin{array}{c} \mathbf{x_B} \\ \mathbf{0} \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right] \tag{47}$$

from which

$$\mathbf{x_B} = \mathbf{B^{-1} b} \tag{48}$$

Equations (47) and (48) represent genuine corner which is feasible provided $\mathbf{x_B} \geq \mathbf{0}$. The objective function $\mathbf{cx}$ at this corner has the value

$$\mathbf{cx} = \left[\begin{array}{c|c} \mathbf{c_B} & \mathbf{c_N} \end{array}\right] \left[\begin{array}{c} \mathbf{x_B} \\ \mathbf{0} \end{array}\right] = \mathbf{c_B x_B} = \mathbf{c_B B^{-1} b} \tag{49}$$

Multiplication of equation (47) from the left by $\mathbf{B^{-1}}$ reduces the square part $\mathbf{B}$ to the identity matrix

$$\left[\begin{array}{c|c} \mathbf{I} & \mathbf{B^{-1} N} \end{array}\right] \left[\begin{array}{c} \mathbf{x_B} \\ \mathbf{0} \end{array}\right] = \left[\begin{array}{c} \mathbf{B^{-1} b} \end{array}\right] \tag{50}$$

If zero components of $\mathbf{x}$ increase to some values $\mathbf{x_N}$ the nonzero components $\mathbf{x_B}$ must drop by $\mathbf{B^{-1} N x_N}$ to maintain equality in (50). That changes the value of the objective function $\mathbf{cx}$ from $\mathbf{c_B x_B}$ given in equation (49) to

$$\mathbf{cx} = \mathbf{c_B} \left(\mathbf{x_B} - \mathbf{B^{-1} N x_N}\right) + \mathbf{c_N x_N} \tag{51}$$

After rearranging this into

$$\mathbf{cx} = \left(\mathbf{c_N} - \mathbf{c_B B^{-1} N}\right) \mathbf{x_N} + \mathbf{c_B x_B} \tag{52}$$

we can see whether the objective function goes up or down as $\mathbf{x_N}$ increases. Everything depends on the sign of the vector $\mathbf{r}$ in parentheses:

$$\mathbf{r} = \mathbf{c_N} - \mathbf{c_B B^{-1} N} \tag{53}$$

In our example the goal is to maximize the objective function $\mathbf{cx}$. If $\mathbf{r} \leq \mathbf{0}$ the current corner is optimal. The product $\mathbf{r x_N}$ cannot be greater than zero since $\mathbf{x} \geq 0$, so the best decision is to keep $\mathbf{x_N} = \mathbf{0}$ and stop. On the other hand, suppose a component of $\mathbf{r}$ is greater than zero. Then by increasing the appropriate corresponding component of $\mathbf{x_N}$ in $\mathbf{rx_N}$ the objective function becomes increased. The simplex method chooses one variable x_i from the vector of **nonbasic** variables $\mathbf{x_N}$—normally the one variable with the most positive component of $\mathbf{r}$—and allows it to become a **basic** variable. In other words, the decision of choosing a new **basic** variable is made by computing the expression for the product $\mathbf{rx_N}$ and by finding out which component of $\mathbf{x_N}$ increases the objective function $\mathbf{cx}$ at most.

In the example the product $\mathbf{rx_N}$ becomes

$$\mathbf{rx_N} = \left(\mathbf{c_N} - \mathbf{c_B}\mathbf{B^{-1}N}\right)\mathbf{x_N} =$$

$$= \left(\begin{bmatrix} 5 & 4 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 5x_1 + 4x_2 + 3x_3 \quad (54)$$

Which component of $\mathbf{x_N}$ does increase the objective function at most? Variable x_1 will maximize the new $\mathbf{cx}$ at most, and x_1 will be the variable entering the set of **basic** variables therefore.

After $\mathbf{rx_N}$ is computed and the x_i entering the base is chosen there is one more question. Which component x_j should leave the set of **basic** variables and become a **nonbasic** variable? If $\mathbf{x_N}$ become incresed from zero then instead of equation

$$\begin{bmatrix} \mathbf{I} & | & \mathbf{B^{-1}N} \end{bmatrix} \begin{bmatrix} \mathbf{x_B} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B^{-1}b} \end{bmatrix} \quad (55)$$

we get

$$\begin{bmatrix} \mathbf{I} & | & \mathbf{B^{-1}N} \end{bmatrix} \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \begin{bmatrix} \mathbf{B^{-1}b} \end{bmatrix} \quad (56)$$

what is equivalent to

$$\mathbf{Ix_B} + \mathbf{B^{-1}Nx_N} = \mathbf{B^{-1}b} \quad (57)$$

In the example the equation (57) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{58}$$

what is

$$\begin{array}{rrrrrrl} x_4 & & & + 2x_1 & + 3x_2 & + \ \ x_3 & = \ \ 5 \\ & x_5 & & + 4x_1 & + \ \ x_2 & + 2x_3 & = 11 \\ & & x_6 & + 3x_1 & + 4x_2 & + 2x_3 & = \ \ 8 \end{array} \tag{59}$$

Having **nonbasic** variables $x_2 = x_3 = 0$ and trying to increase **nonbasic** x_1 above zero we get

$$\begin{array}{l} x_4 + 2x_1 = \ \ 5 \\ x_5 + 4x_1 = 11 \\ x_6 + 3x_1 = \ \ 8 \end{array} \tag{60}$$

Because $\mathbf{x_B} \ \geq \ \mathbf{0}$ we have $x_4, x_5, x_6 \ \geq \ 0$

$$\begin{array}{l} 0 \ \leq \ x_4 = \ \ 5 - 2x_1 \\ 0 \ \leq \ x_5 = 11 - 4x_1 \\ 0 \ \leq \ x_6 = \ \ 8 - 3x_1 \end{array} \tag{61}$$

what leads to inequalities for **nonbasic** variable x_1 which we try to increase

$$\begin{array}{l} x_1 \ \leq \ \frac{5}{2} \\ x_1 \ \leq \ \frac{11}{4} \\ x_1 \ \leq \ \frac{8}{3} \end{array} \tag{62}$$

From inequalities (62) it can be seen how large **nonbasic** x_1 can become. The currently **basic** variable x_4 allows x_1 to increase only to $\frac{5}{2}$ and therefore x_4 is the worst **basic** component from the point of view of increasing **nonbasic** variable x_1. The goal was to increase **nonbasic** x_1 as much as possible and move it into the **base**. Because the **basic** x_4 stands in the way of that at most it will become removed from the **base**.

In general, after finding a good **nonbasic** candidate variable x_i to enter the **base** one has to determine which **basic** candidate variable x_j is the best to exit the base. It will be this **basic** variable x_j which prevents the **nonbasic** candidate variable x_i from growing above the smallest positive value among

$$\frac{(\mathbf{B^{-1}b})_k}{(\mathbf{B^{-1}N})_{ki}} \tag{63}$$

where $(\mathbf{B^{-1}N})_{ki}$ is the element of matrix $\mathbf{B^{-1}N}$ from k-th row and i-th column, and $(\mathbf{B^{-1}b})_k$ is the k-th component of vector $\mathbf{B^{-1}b}$. Index k runs over all possible values of the ratio (63) and corresponds to indexes of all current **basic** variables. Among those **basic** variables there will be a **basic** variable x_j for which the j-th ratio (63) has the smallest positive value limiting the growth of the **nonbasic** variable x_i which needs to be increased as much as possible. Because the **basic** variable x_j is the strongest reason for limiting the growth of **nonbasic** variable x_i, the **basic** variable x_j will leave the base. Of course, as previously established, the **nonbasic** variable x_i will enter the base.

In the example at the beginning the set of **basic** variables was $\mathcal{S}_B = \{x_4, x_5, x_6\}$ and the set of **nonbasic** variables was $\mathcal{S}_N = \{x_1, x_2, x_3\}$. After the first step of simplex method the variable x_4 leaves the base, and the variable x_1 enters the base. In the second step of simplex method the set of **basic** variables is $\mathcal{S}_B = \{x_1, x_5, x_6\}$ and the set of **nonbasic** variables is $\mathcal{S}_N = \{x_2, x_3, x_4\}$. Like at the very beginning now the equation

$$\begin{array}{l} 2x_1 + 3x_2 + x_3 + x_4 = 5 \\ 4x_1 + x_2 + 2x_3 + x_5 = 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 = 8 \end{array} \tag{64}$$

needs to be solved with with **nonbasic variables** equal to zeros. This time one needs to put $x_2 = x_3 = x_4 = 0$ and solve

$$\mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{b} \tag{65}$$

which is

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \tag{66}$$

Now

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \tag{67}$$

and its inverse is

$$\mathbf{B^{-1}} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} \tag{68}$$

With the choice $\mathbf{x_N} = \mathbf{0}$

$$\mathbf{x_B} = \mathbf{B^{-1}b} \tag{69}$$

gives the solution

$$\mathbf{x_B} = \begin{bmatrix} x_1 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ 1/2 \end{bmatrix} \tag{70}$$

The solution represents a simplex corner $(x_1, x_2, x_3, x_4, x_5, x_6)$ with coordinates $(\mathbf{5/2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1/2})$. The objective function at this point has the value

$$\mathbf{cx} = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5/2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1/2 \end{bmatrix} = 25/2 \tag{71}$$

what is an improvement in comparison to the value $\mathbf{cx} = 0$ at previous point $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{5}, \mathbf{11}, \mathbf{8})$. For the set of **basic** variables $\mathcal{S}_B = \{x_1, x_5, x_6\}$ and the set of **nonbasic** variables is $\mathcal{S}_N = \{x_2, x_3, x_4\}$ the row vectors of coefficients of objective function are

$$\mathbf{c_B} = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \tag{72}$$

and

$$\mathbf{c_N} = \begin{bmatrix} 4 & 3 & 0 \end{bmatrix} \tag{73}$$

for **basic** and **nonbasic** variables, respectively. At this step in the example the product $\mathbf{rx_N}$ becomes

$$\mathbf{rx_N} = \left(\mathbf{c_N} - \mathbf{c_B B^{-1} N}\right) \mathbf{x_N} =$$

$$= \left(\begin{bmatrix} 4 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

$$= \begin{bmatrix} -7/2 & 1/2 & -5/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = -\frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \quad (74)$$

From equation (74) one can conclude that the **nonbasic** variable x_3 is the best candidate to enter the base in the next step of simplex algorithm because it can increase the objective function at most. Looking at the equation

$$\mathbf{I}\mathbf{x_B} + \mathbf{B^{-1}}\mathbf{N}\mathbf{x_N} = \mathbf{B^{-1}}\mathbf{b} \quad (75)$$

it can be decided which **basic** variable shall leave the base. The matrix product $\mathbf{B^{-1}N}$ is

$$\mathbf{B^{-1}N} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 & 1/2 \\ -5 & 0 & -2 \\ -1/2 & 1/2 & -3/2 \end{bmatrix} \quad (76)$$

and the equation (75) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 3/2 & 1/2 & 1/2 \\ -5 & 0 & -2 \\ -1/2 & 1/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ 1/2 \end{bmatrix} \quad (77)$$

what now is

$$\begin{aligned} x_1 \qquad\qquad & + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 = \tfrac{5}{2} \\ x_5 \qquad & - 5x_2 \qquad\qquad - 2x_4 = 1 \\ x_6 & + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - \tfrac{3}{2}x_4 = \tfrac{1}{2} \end{aligned} \quad (78)$$

Having **nonbasic** variables $x_2 = x_4 = 0$ and trying to increase **nonbasic** x_3 above zero we get

$$\begin{aligned} x_1 + \tfrac{1}{2}x_3 &= \tfrac{5}{2} \\ x_5 \qquad &= 1 \\ x_6 + \tfrac{1}{2}x_3 &= \tfrac{1}{2} \end{aligned} \quad (79)$$

Because $\mathbf{x_B} \geq \mathbf{0}$ we have $x_1, x_5, x_6 \geq 0$

$$\begin{aligned} 0 \leq\ & x_1 = \tfrac{5}{2} - \tfrac{1}{2}x_3 \\ 0 \leq\ & x_5 = 1 \\ 0 \leq\ & x_6 = \tfrac{1}{2} - \tfrac{1}{2}x_3 \end{aligned} \quad (80)$$

what leads to conditions for **nonbasic** variable x_3 which we try to increase as much as possible

$$\begin{aligned} x_3 &\leq 5 \\ x_3 &= 1 \\ x_3 &\leq 1 \end{aligned} \tag{81}$$

One can see that the condition $x_3 \leq 1$ derived from the inequality involving **basic** variable x_6 is the most limiting condition put upon the growth of the **nonbasic** variable x_3. Again, from inequalities and equation (81) it can be seen how large **nonbasic** x_3 can become. The currently **basic** variable x_6 allows x_3 to increase only to 1 and therefore x_6 is the worst **basic** component from the point of view of increasing **nonbasic** variable x_3. The goal was to increase **nonbasic** x_3 as much as possible and move it into the **base**. Because the **basic** x_6 stands in the way of that at most it will become removed from the **base**.

In the current step of simplex method the set of **basic** variables was $\mathcal{S}_B = \{x_1, x_5, x_6\}$ and the set of **nonbasic** variables was $\mathcal{S}_N = \{x_2, x_3, x_4\}$. For the next step x_3 enters the base, and x_6 leaves the base what leads to the new set of **basic** variables $\mathcal{S}_B = \{x_1, x_3, x_5\}$ and the set of **nonbasic** variables $\mathcal{S}_N = \{x_2, x_4, x_6\}$.

The third step of simplex algorithm begins. We can use equation (77) from previous step to rearrange and separate **basic** and **nonbasic** variables in order to obtain new matrices $\mathbf{B}$ and $\mathbf{N}$ what leads us to the system

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} + \begin{bmatrix} 3/2 & 1/2 & 0 \\ -5 & -2 & 0 \\ -1/2 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ 1/2 \end{bmatrix} \tag{82}$$

This time

$$\mathbf{B} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \tag{83}$$

and its inverse is

$$\mathbf{B^{-1}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \tag{84}$$

The choice of $\mathbf{x_N} = \mathbf{0}$

$$\mathbf{x_B} = \mathbf{B^{-1}b} \tag{85}$$

gives the solution

$$\mathbf{x_B} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5/2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{86}$$

The solution represents a simplex corner $(x_1, x_2, x_3, x_4, x_5, x_6)$ with coordinates $(\mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0})$. The objective function at this point has the value

$$\mathbf{cx} = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 13 \tag{87}$$

what again is a higher and therefore a better value of objective function than the value 25/2 from the previous step. The row vectors of coefficients of objective function now are

$$\mathbf{c_B} = \begin{bmatrix} 5 & 3 & 0 \end{bmatrix} \tag{88}$$

and

$$\mathbf{c_N} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \tag{89}$$

for **basic** and **nonbasic** variables, respectively. At the third step in the example the product $\mathbf{rx_N}$ becomes

$$\mathbf{rx_N} = \left(\mathbf{c_N} - \mathbf{c_B B^{-1} N}\right) \mathbf{x_N} =$$

$$\left(\begin{bmatrix} 4 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3/2 & 1/2 & 0 \\ -5 & -2 & 0 \\ -1/2 & -3/2 & 1 \end{bmatrix} \right) \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} = -3x_2 - x_4 - x_6 \tag{90}$$

From the equation (90) it can be seen that any increase in value of **nonbasic** variables x_2, x_4 or x_6 will decrease the objective function. Optimization must stop here then.

References

[1] Gilbert Strang (1986) *Introduction to Applied Mathematics* Wellesley-Cambridge Press

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